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A Cycle Index Sum Inversion Theorem

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A frequently encountered cycle index sum equation is one of the form $Z(W)[Z(T)] = Z(C)$, where $Z(C)$ and $Z(T)$ are well known and where the generating function $W(x)$ is desired. $W(x)$ is most efficiently computed by inverting the above equation to $W(x) = Z(C)[M(x)]$. It is shown in this paper that the generating function $M(x)$ which performs this inversion has a simple combinatorial description.

INTRODUCTION

R. W. Robinson's Composition Theorem successfully deals with the problem of enumerating a set of graphs W which is related to an easily enumerable set of graphs C by the relation that the graphs in the set C are obtained by planting point rooted graphs from a set T on the points of the graphs in W. An example of this is to let W be the set of all connected graphs without endpoints excepting the single point, to let C be the set of all connected graphs which are not trees, and to let T be the set of all rooted trees. Clearly, each graph in C can be obtained from a unique graph α in W by placing rooted trees on the points of α. The composition theorem gives

$$Z(C) = Z(W)[Z(T)], \tag{*}$$

where $Z(W)[Z(T)]$ is the composition product

$$Z(W)[Z(T)] = Z(W)[s_i \to Z(T)[s_j \to s_{ij}]].$$

Equation (*) is typical of equations that come up in such an enumeration in that the cycle index of the set to be counted sits to the inside of the composition product. One can recover $Z(W)$ from (*) by a simple comparison of coefficiens, but computationally such a procedure is very clumsy. In the first place, $Z(W)$ must be stored during computation and in the second place, the eventual outcome is $Z(W)$. For enumeration purposes,

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the ordinary generating function $W(x)$ of the set W is sufficient; $Z(W)$ includes much more information than is needed.

To avoid these inefficiencies, (*) should be inverted to an equation of the form

$$Z(C)[Z(M)] = Z(W) \qquad (**)$$

and then simplified to

$$Z(C)[M(x)] = W(x) \qquad (***)$$

by sending x_i to x^i. When $W(x)$ is computed using (***), no cycle indices need be stored and no more information than the ordinary generating function is produced. Computationally (***) is very good.

The question arises as to what power series $M(x)$ inverts (*) to (***). R. W. Robinson has recently developed algebraic techniques which can be used to solve for $M(x)$. This paper gives a different, more combinatorial solution in which the answer comes out in terms of the atoms of a certain partial ordering associated with the set T.

It will be assumed throughout that the reader is familiar with the content of R. W. Robinson's paper *Enumeration of Non-Separable Graphs* [2].

1. Preliminaries

Let $(P, \leqslant)$ be a finite, partially ordered set. Recall that the Möbius function, μ, is an integer-valued function defined on the intervals of $(P, \leqslant)$ by

$$\begin{aligned} \mu(p, r) &= 1 && \text{if} \quad p = r \\ &= -\sum_{p \leqslant s < r} \mu(p, s) && \text{if} \quad p < r. \end{aligned}$$

By the definition of μ, it is clear that the sum of the Möbius function over an interval $p \leqslant r$ is 0 of $p < r$ and 1 if $p = r$.

The following notational conventions will be used. Sets of labelled graphs will be denoted by small Roman letters and the corresponding sets of isomorphism classes of graphs by the corresponding capital Roman letters. If u and U are two such sets of graphs, $u(x)$ will denote the exponential generating function for the set u and $U(x)$ will denote the ordinary generating function for the set U. $Z(U)$ will denote the cycle index sum for the set U. Cycle indices will be taken with respect to points only; the variable s_n will denote an n-cycle of points.

In the case where the graphs in the set U are unrooted, $Z(U')$ will denote

$\partial Z(U)/\partial s_1$. By a well-known result of R. Z. Norman [1], $Z(U')$ is the cycle index sum taken over all rooted graphs whose unrooted versions lie in U.

A canonically labelled graph of order k is a pair $(V(g), E(g))$, where $V(g) = \{1, 2,..., k\}$ and where $E(g)$ is a set of 2-element subsets of $V(g)$. No loops or multiple edges will be allowed. The reader is warned that there will also be the notion of an *n-labelled graph of order k*. In general, these will not be canonically labelled graphs.

We say that two canonically labelled graphs g and h, both of order k, are *isomorphic* if there exists a permutation $\sigma \in S_k$ such that

$$E(h)\,\sigma = \{\{i\sigma, j\sigma\}: \{i,j\} \in E(h)\} = E(g).$$

An *unlabelled graph of order* k is an isomorphism class of canonically labelled graphs of order k.

2. The Basic Counting Theorem

In this section we blend together Burnside's Lemma and the Möbius Inversion Theorem to produce a tool which will be used in Section 3 to prove the main theorem.

Suppose $(P, \leqslant)$ is a finite partially ordered set and G is a subgroup of its automorphism group. Let $P(G)$ denote the set of orbits of P under the action of G. We obtain a partial ordering $\leqslant_G$ on $P(G)$ by defining for orbits X and Y, $X \leqslant_G Y$ if there exist $x \in X$, $y \in Y$ with $x \leqslant y$. Our interest in this section will be how to Möbius invert over the partial ordering $P(G)$. The partial ordering $(P(G), \leqslant_G)$ is called the *condensation* of $(P, \leqslant)$ mod G.

The idea used in Theorem 1 is quite simple. For each $\sigma \in G$, let P_σ denote the partial ordering of points fixed by σ together with the inherited ordering. To effect a kind of Möbius inversion on the partial ordering $P(G)$ of orbits we mimic Burnside's Lemma; i.e., we invert within each fixed point partial ordering P_σ and then average the results over the group G.

With notation as above, let $F(G; P)$ be the subset of $G \times P$ defined by

$$F(G; P) = \{(\sigma, p): p\sigma = p\},$$

and let R be a commutative ring containing $\mathbb{Q}$. Let $\gamma: F(G; P) \to R$ satisfy

$$\gamma(\sigma, p) = \gamma(\eta^{-1}\sigma\eta, p\eta) \tag{2.1}$$

$\forall(\sigma, p) \in F(G; P)$, $\forall \eta \in G$. For each orbit $X \subseteq P$, define a weight $Z_\gamma(X) \in R$ by

$$Z_\gamma(X) = \frac{1}{|G_x|} \sum_{\sigma \in G_x} \gamma(\sigma, x) \tag{2.2}$$

for arbitrarily chosen $x \in X$. The conjugacy condition (2.1) imposed on γ assures us that the right side of (2.2) is independent of the choice of x from X.

For $\sigma \in G$, let μ_{P_σ} denote the Möbius function within the partial ordering P_σ.

THEOREM 1. *Let notation be as above. Suppose in addition that for each $x \in P$*

(A) *there exists a unique minimal $m(x) \leqslant x$;*

(B) *if x is stabilized by $\sigma \in \Gamma(P)$ then $m(x)$ is also stabilized by σ. Then*

$$\sum_{\substack{X \in P(G) \\ X \text{ minimal}}} Z_\gamma(X) = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{x \in P_\sigma} \Bigg(\sum_{\substack{y \in P_\sigma \\ y \leqslant x}} \mu_{P_\sigma}(y, x)\, \gamma(\sigma, x) \Bigg).$$

Proof.

$$\begin{aligned} &\frac{1}{|G|} \sum_{\sigma \in G} \sum_{x \in P_\sigma} \Bigg(\sum_{\substack{y \in P_\sigma \\ y \leqslant x}} \mu_{P_\sigma}(y, x)\, \gamma(\sigma, x) \Bigg) \\ &\quad = \sum_{x \in P} \frac{1}{|G|} \sum_{\sigma \in G_x} \Bigg(\sum_{\substack{y \in P_\sigma \\ y \leqslant x}} \mu_{P_\sigma}(y, x) \Bigg) \gamma(\sigma, x). \end{aligned} \tag{2.3}$$

For each $x \in P$, let X denote the orbit containing x. By Burnside's Lemma we have $|G| = |X|\,|G_x|$ and so (2.3) can be rewritten as

$$\sum_{X \in P(G)} \frac{1}{|X|} \sum_{x \in X} \Bigg(\frac{1}{|G_x|} \sum_{\sigma \in G_x} \Bigg(\sum_{\substack{y \in P_\sigma \\ y \leqslant x}} \mu_{P_\sigma}(y, x) \Bigg) \gamma(\sigma, x) \Bigg). \tag{2.4}$$

Fix attention on $x \in P$ and $\sigma \in G_x$. Then if x is not minimal,

$$\sum_{\substack{y \in P_\sigma \\ y \leqslant x}} \mu_{P_\sigma}(y, x)$$

is the sum of the Möbius function of P_σ over the proper interval $(m(x), x)$. So

$$\sum_{\substack{y \in P_\sigma \\ y \leqslant x}} \mu_{P_\sigma}(y, x) = \begin{cases} 0 & \text{if } x \text{ is not minimal} \\ 1 & \text{if } x \text{ is minimal.} \end{cases}$$

Hence we can rewrite (2.4) as

$$\sum_{\substack{X \in P(G) \\ X \text{ minimal}}} \frac{1}{|X|} \sum_{x \in X} Z_y(X) = \sum_{\substack{x \in P(G) \\ X \text{ minimal}}} Z_y(X). \quad \blacksquare$$

3. The Main Result

We are now in a position to answer the question of what generating function $M(x)$ will invert the equation

$$Z(C) = Z(W)[Z(T)] \tag{*}$$

to the equation

$$W(x) = Z(C)[M(x)]. \tag{***}$$

The answer will be given in terms of a group-condensed partial ordering on the set T. We begin by defining that partial ordering.

Definition 3.1. Let n be a positive integer. An *n-labelled graph* g of order k is a pair $(V(g), E(g))$, where

(i) $V(G) \subseteq \{1, 2,..., n\}$ and $|V(g)| = k$,
(ii) $E(g)$ is a set of 2-element subsets of $V(g)$.

Clearly a canonically labelled graph of order k is an n-labelled graph for each $n \geqslant k$. Associated with each n-labelled graph g of order k is a canonically labelled graph $\bar{g}$ of order k which is given by assigning the numbers $\{1, 2,..., k\}$ to the vertices of g in such a way as to respect the natural ordering of the vertices. In Fig. 1 we see g and $\bar{g}$ for g a 15-labelled graph. Clearly, for each canonically labelled graph h of order k there exist $\binom{n}{k}$ n-labelled graphs g of order k with $\bar{g} = h$.

If s is a set of canonically labelled graphs and n is a positive integer, we let s_n denote the set of n-labelled graphs g with $\bar{g} \in s$. Let $s_\infty = \bigcup_{n=1}^{\infty} s_n$. The elements of s_∞ are called *labelled graphs in s*.

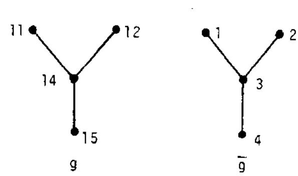


Figure 1

DEFINITION 3.2. Let c be the set of all rooted, connected, canonically labelled graphs, and let $w, x \in c_\infty$. We say w is obtained from x by *vertex substitution* if there exists a set of rooted, connected labelled graphs

$$\{y_i : i \in V(x)\} \subseteq c_\infty$$

such that

(1) the root point of y_i is labelled i;

(2) if $i \neq j$ then y_i and y_j have no points with a common label;

(3) $V(w) = \bigcup_{i \in V(x)} V(y_i)$, $E(w) = E(x) \cup (\bigcup_{i \in V(x)} E(y_i))$;

(4) the root point of w has the same label as the root point of x.

All this definition says is that w is obtained from x by replacing each point i of x by the root point of y_i. For example, let

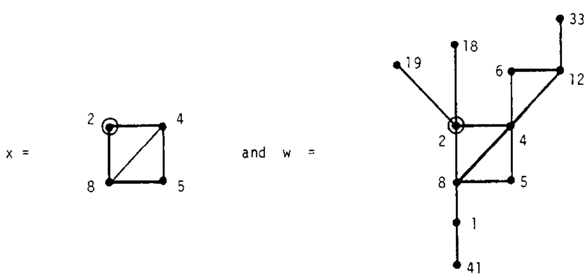


FIGURE 2

then w is obtained from x by vertex substitution, by letting

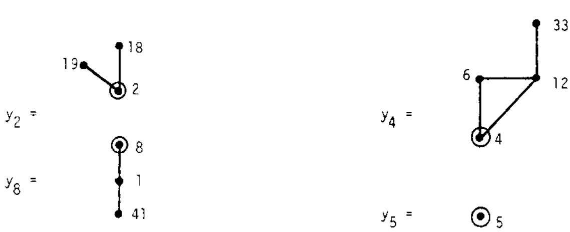


FIGURE 3

DEFINITION 3.3. A set t of rooted, connected, canonically labelled graphs is *closed under vertex substitution* provided that if x is a graph, if $\{y_i : i \in V(x)\} \subseteq t_\infty$ and if w is obtained from x by vertex substitution with the graphs y_i, then $x \in t_\infty$ iff $w \in t_\infty$. If the set t is closed under vertex substitution and contains the 1-point rooted graph then t is called *treelike*.

Examples of treelike sets are

(i) all rooted connected, canonically labelled graphs,

(ii) all rooted trees,

(iii) the 1-point graph together with all rooted, connected, canonically labelled graphs whose blocks come from some fixed set D.

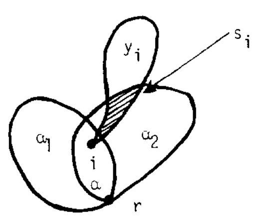
Let t be a treelike set of rooted, connected, canonically labelled graphs. Define the partial ordering $\leqslant$ on t_∞ by $x \leqslant w$ if w is obtained from x by vertex substitution. Using this partial ordering on t_∞ we obtain a partial ordering $\leqq$ on T by saying $X \leqq W$ if there exist labellings $w, x \in t_\infty$ of W and X such that $x \leqslant w$. It is important to note that this partial ordering on T can be obtained by group condensing the partial ordering $(t_n, \leqslant) \bmod S_n$ and letting $n \to \infty$. Here $(t_n, \leqslant)$ refers to the restriction of $\leqslant$ to the set t_n. The partial ordering $(T, \leqq)$ will be called the *composition ordering of* T.

PROPOSITION 1. *Let t be a treelike set of rooted, connected, labelled graphs. Let $x \in t_\infty$ and let r denote the root point of x. Suppose that $x - \{r\}$ is connected. Then there is a unique atom in the closed interval $[r, x]$ of $(t_\infty, \leqslant)$.*

Proof. Assume to the contrary that α_1 and α_2 are distinct atoms below x having root point r. Let α denote the subgraph of x induced by $V(\alpha_1) \cap V(\alpha_2)$. Note that α contains more than the point r since $x - \{r\}$ is connected.

Since $\alpha_1 \leqslant x$ we know that x is obtained from α_1 by vertex substitution with graphs in t_∞. Let x be obtained by the substitution $\{y_i : i \in V(\alpha_1)\}$. Similarly, let x be obtained from α_2 by the substitution $\{z_j : j \in V(\alpha_2)\}$. Note that $y_i, z_j \in t_\infty$ for all i, j.

For each $i \in V(\alpha)$, let s_i denote the subgraph of α_2 induced by $V(y_i) \cap V(\alpha_2)$.



Note that y_i is obtained from s_i by vertex substitution with the graphs $\{z_j : j \in s_i\}$ and so $s_i \in t_\infty$ since $y_i \in t\infty$. Hence α_2 is obtained from α by the vertex substitution $\{s_i : i \in V(\alpha)\}$, where each $s_i \in t_\infty$. So $\alpha \in t_\infty$ since $\alpha_2 \in t_\infty$, and $\alpha \leqslant \alpha_2$. We see $\alpha = \alpha_2$ as α_2 is an atom and $\alpha > r$.

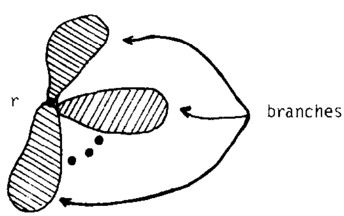
Similarly $\alpha_1 = \alpha$ and so $\alpha_1 = \alpha_2$.

Let t be a treelike set of rooted, connected, canonically labelled graphs. A graph $x \in t_\infty$ with root point r is an *atom* if the interval $[r, x]$ contains only x and r. The set t is called *unrefineable* if $x - \{r\}$ is connected for each atom $x \in t_\infty$. T is *unrefineable* if t is unrefineable. In this section we prove the main theorem in the case that T is unrefineable.

LEMMA 1. *Let T be an unrefineable set of rooted, connected graphs and let $\mathcal{A}$ be the set of atoms in its associated composition ordering. Then*

$$\lim_{n\to\infty} \frac{1}{n!} \left(\sum_{\sigma \in S_n} \sum_{w \in (t_n)_\sigma} \mu_{(t_n)_\sigma}(0, w)\, x^{|w|} \right) = x \left(\prod_{A \in \mathcal{A}} (1 - x^{|V(A)|-1}) \right).$$

Proof. Let $w \in t_n$ and let r be its root point. The subgraphs of w induced by r together with a connected component of $w - \{r\}$ will be called the branches of w. Let $\sigma \in S_n$ and let $w \in (t_n)_\sigma$. Then the action of σ on w



induces an action of σ on the branches of w. So for fixed n, fixed $\sigma \in S_n$ and fixed nonzero $w \in (t_n)_\sigma$, define cycles

$$\begin{gathered} B_1 = (b_{1,1}, \ldots, b_{1,l_1}) \\ \vdots \\ B_k = (b_{k,1}, \ldots, b_{k,l_k}) \end{gathered}$$

by letting the $b_{i,j}$ be the branches of w and by letting $B_1, \ldots, B_k$ be the cycles in the disjoint cycle decomposition of σ considered as a permutation of the branches.

Observe that β_i, the subgraph induced by B_i is invariant under σ; so $\beta_i \in (t_n)_\sigma$ for each i, since t is unrefineable. The interval from 0 to w in $(t_n)_\sigma$ is isomorphic as a partially ordered set to the direct product over i of the intervals from 0 to β_i in $(t_n)_\sigma$. Hence

$$\mu_{(t_n)_\sigma}(0, w) = \mu_{(t_n)_\sigma}(0, \beta_1) \cdots \mu_{(t_n)_\sigma}(0, \beta_k). \tag{3.1}$$

The next observation is that

$$\begin{aligned} \mu_{(t_n)_\sigma}(0, w) &= 0 && \text{if } b_{i,j} \text{ fails to be an atom in } t_n \text{ for any } i, j, \\ &= (-1)^k && \text{if each } b_{i,j} \text{ is an atom in } t_n. \end{aligned}$$

To see this, suppose $b_{i,j}$ is not an atom in t_n for some i and j. Hence $v \leqslant b_{i,j}$ for some v in t_n which is an atom. Then

$$\begin{gathered} v\sigma \leqslant b_{i,j+1}, \\ v\sigma^2 \leqslant b_{i,j+2} \\ \vdots \\ v\sigma^{l_i-1} \leqslant b_{i,j-1} \end{gathered}$$

and each $v\sigma^m$ is an atom in t_n (as each $\sigma \in S_n$ is a automorphism of t_n). Let η_i be the subgraph of β_i induced by the union of $v, v\sigma, \ldots, v\sigma^{l_i-1}$. Clearly $\eta_i \in (t_n)_\sigma$ and $\eta_i < \beta_i$. As each $v\sigma^m$ is an atom in t_n and as σ acts transitively on the set of $v\sigma^m$, η_i is an atom in $(t_n)_\sigma$. By Proposition 1, η_i is the unique atom in the interval $(0, \beta_i)$, where the ordering is the ordering inherited from $(t_n)_\sigma$. Hence, by the Crosscut Theorem (see Rota [3]),

$$\mu_{(t_n)_\sigma}(0, \beta_i) = 0.$$

On the other hand, if each $b_{i,j}$ is an atom in t_n then $\beta_i = \eta_i$ for each i and so

$$\mu_{(t_n)_\sigma}(0, \beta_i) = -1.$$

In view of (3.1) this verifies the last observation.

Next observe that for each i, all the $b_{i,j}$'s are isomorphic as rooted graphs (via σ). So if w contributes a nonzero term to

$$\frac{1}{n!} \sum_{\sigma \in S_n} \sum_{w \in (t_n)} \mu_{(t_n)_\sigma}(0, w)\, x^{|w|}, \tag{3.2}$$

then we can choose distinct isomorphism classes of atoms $A_1, \ldots, A_k$ in t_n and positive integers $l_1, \ldots, l_k$ such that w consists of l_1 atoms from class A_1, l_2 atoms from class $A_2, \ldots,$ and l_k atoms from class A_k, where

$$1 + \sum_{i=1}^{k} l_i(|V(A_i)| - 1) = s \leqslant n.$$

Conversely, consider the contribution to (3.2) made by any w constructed from atoms as above. Such a w has as automorphism group the direct product $G_1 \otimes G_2 \otimes \cdots \otimes G_k$ of the k groups defined by

$$G_i \cong (\Gamma(A_i) \text{ wreathed over } S_{l_i}).$$

So the order of the automorphism group of w is

$$(l_1!\, |\Gamma(A_1)|^{l_1} \cdots l_k!\, |\Gamma(A_k)|^{l_k}). \tag{3.3}$$

Hence, such a w will contribute to (3.2) a factor of

$$\begin{aligned} &((n-s)!/n!)(l_1!\,|\Gamma(A_1)|^{l_1} Z(S_{l_1})[s_i \to -Z(A_1)[s_j \to x^{ij}]]) \\ &\cdots (l_k!\,|\Gamma(A_k)|^{l_k} Z(S_{l_k})[s_i \to -Z(A_k)[s_i \to x^{ij}]]). \end{aligned} \tag{3.4}$$

In the above expression, the cycle index terms

$$Z(S_{l_u})[s_i \to -Z(A_u)[s_j \to x^{ij}]] \tag{3.5}$$

give the cycle index sum of $\Gamma(A_u)$ wreathed over S_{l_u} with x^v substituted for s_v and with a factor of -1 included for each cycle of atoms from A_u. The latter factor accounts for the Möbius function.

The cycle index begins with a factor of one over the order of the group which is not included in the sum (3.2). So it is necessary to multiply each of these cycle index terms (3.5) by the order of the wreath product group. Hence the factors $l_u!\,|\Gamma(A_u)|^{l_u}$ appear in (3.4). The factor $(n-s)!$ is included in (3.4) to account for the number of ways the permutation σ, which is now determined on w, can be extended to a permutation of $\{1, 2,..., n\}$ (observe that the permutation in (3.2) is taken from S_n and that w has only s points, where s may be less than n).

Factoring out x in (3.4) we obtain

$$\begin{aligned} &((n-s)!/n!)\, x^s(l_1!\,|\Gamma(A_1)|^{l_1} \cdots l_k!\,|\Gamma(A_k)|^{l_k})\, Z(S_{l_1})[s_i \to -1] \\ &\qquad \cdots Z(S_{l_k})[s_i \to -1]. \end{aligned} \tag{3.6}$$

It is easy to show that

$$\begin{aligned} Z(S_l)[s_i \to -1] &= 1 \quad \text{if } l = 0 \\ &= -1 \quad \text{if } l = 1 \\ &= 0 \quad \text{if } l > 1 \end{aligned}$$

and so w contributes 0 to (3.2) unless w consists of k atoms all chosen from distinct isomorphism classes $A_1,..., A_k$. In that case, the contribution of w is

$$(x^s(n-s)!/n!)(|\Gamma(A_1)| \cdots |\Gamma(A_k)|)(-1)^k. \tag{3.3}$$

Recall that the number of ways to label a graph on n points with the numbers $\{1, 2,..., n\}$ is $n!$ divided by the order of the automorphism group of that graph. Using this fact, it is clear that the number of ways to choose k representatives, one from each of a fixed set of atoms $A_1,..., A_k$, is

$$\binom{n}{|V(A_1)|, \cdots |V(A_k)|, n-s} \left(\frac{|V(A_1)|!}{|\Gamma(A_1)|}\right) \cdots \left(\frac{|V(A_k)|!}{|\Gamma(A_k)|}\right).$$

So, (3.6) summed over all w built from an atom out of each of the classes $A_1,\ldots,A_k$ collapses to

$$x^{|V(A_1)|} \cdots x^{|V(A_k)|}(-1)^k.$$

The result follows immediately. ∎

We now come to the main result of this section; a theorem which inverts (*) to (***) in the case that T is treelike. (T is called *treelike* if t is treelike.)

THEOREM 2A (Unrefineable Orderings). *Let T be an unrefineable treelike set of rooted, connected graphs and let $\mathcal{A}$ be the set of atoms in its associated composition partial ordering. Define $M(x)$ by*

$$M(x) = x\left(\prod_{A\in\mathcal{A}} (1 - x^{|V(A)|-1})\right).$$

Suppose C and W are sets of graphs satisfying

$$Z(C) = Z(W)[Z(T)].$$

Then

$$W(x) = Z(C)[M(x)].$$

Proof. Without loss of generality, we may assume that C is precisely the set of graphs obtained by taking graphs $\alpha \in W$ and replacing the points of α by graphs in T. Note that when such a replacement is made, we take the automorphism group of the resultant configuration to be the composition stabilizer group—in particular, α is fixed setwise.

For $n > 0$, define a partial ordering $\leqslant$ on c_n by $w \leqslant v$ if v can be obtained from w through vertex replacement with graphs in t_n. This is easily seen to be a partial ordering (as T is treelike) whose minimal elements are precisely the elements of the set w_n.

Let $G = S_n$ act on c_n. It is straightforward to check that G is a subgroup of $\Gamma(c_n)$ and that $(c_n, \leqslant)$ satisfies the conditions set out at the start of Theorem 1.

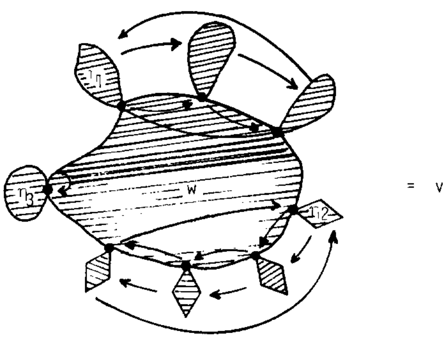
Define $\gamma\colon F(S_n; c_n) \to \mathbb{Q}[x]$ by $\gamma(w,\sigma) = x^{|w|}$. Letting $W(x) = \sum_{i=1}^{\infty} w_i x^i$ and applying Theorem 1 we have

$$w_1 x + \cdots + w_n x^n = \frac{1}{n!} \sum_{\sigma\in S_n} \sum_{v\in(c_n)_\sigma} \left(\sum_{\substack{w\in(c_n)_\sigma \\ w\leqslant v}} \mu_{(c_n)_\sigma}(w,v)\right) x^{|v|}$$

$$= \frac{1}{n!} \sum_{\sigma\in S_n} \sum_{w\in(c_n)_\sigma} \sum_{\substack{v\in(c_n)_\sigma \\ v\geqslant w}} \mu_{(c_n)_\sigma}(w,v)\, x^{|v|}.$$

We now reorganize the above sum by summing first over $w \in c_n$ and over automorphisms $\bar{\sigma}$ of w after which we complete the sum by extending the automorphism $\bar{\sigma}$ to $\{1, 2,..., n\}$ in every possible way.

In doing so we make use of the following observation. Suppose $w, v \in (c_n)_\sigma$ with $w \leqslant v$ and suppose $\sigma|_{V(w)}$ decomposes into disjoint cycles $L_1,..., L_k$ of lengths $l_1,..., l_k$, respectively. Let λ_i be the point of smallest label in L_i and let η_i be the element of t_n which replaces λ_i when v is obtained from w by vertex substitution.



Then the interval from w to v in $(c_n)_\sigma$ is isomorphic as a partially ordered set of the direct product of the intervals $(0, \eta_i)$, where $(0, \eta_i)$ has the order inherited from $(t_n)_{\sigma^{l_i}}$. Observe that $\eta_i \in (t_n)_{\sigma^{l_i}}$ as σ^{l_i} places η_i back on top of itself.

Hence

$$\mu_{(c_n)_\sigma}(w, v)\, x^{|v|} = (\mu_{(t_n)_{\sigma^{l_i}}}(0, \eta)\, x^{|\eta_1| l_1}) \cdots (\mu_{(t_n)_{\sigma^{l_k}}}(0, \eta_k)\, x^{|\eta_k| l_k}).$$

Using this fact and taking limits gives

$$\lim_{n \to \infty} \left(\sum_{i=1}^{n} w_i x^i \right) = \lim_{n \to \infty} Z(C^{(n)}) \left[s_i \to \lim_{m \to \infty} \frac{1}{m!} \sum_{\sigma \in S_m} \sum_{w \in (t_m)_\sigma} (0, w)\, x^{|w| i} \right],$$

where $C^{(n)}$ consists of those graphs in C with n or fewer points. Applying Lemma 1 gives

$$W(x) = Z(C)[M(x)]. \qquad \blacksquare$$

Two things are worth mentioning with regard to Theorem 2. The first is that $M(x)$ may also be written

$$M(x) = x \exp\left(- \sum_{i=1}^{\infty} \frac{\mathcal{O}(x^i)}{i} \right).$$

where $\mathcal{A}(x) = \sum_{A \in \mathcal{A}} x^{|V(A)|-1}$. To see the equivalence, observe that

$$\log\left(\prod_{A \in \mathcal{A}} (1 - x^{|V(A)|-1})\right) = \sum_{A \in \mathcal{A}} \log(1 + (-x^{|V(A)|-1}))$$
$$= -\sum_{A \in \mathcal{A}} \sum_{i=1}^{\infty} \frac{x^{i(|V(A)|-1)}}{i}$$
$$= -\sum_{i=1}^{\infty} \frac{\mathcal{A}(x^i)}{i}.$$

Second, there is a cycle index sum version of Theorem 2A which holds in the unrefineable case.

THEOREM 3. *Let T be an unrefineable treelike set of rooted, connected graphs and let $\mathcal{A}$ be the set of atoms in its associated composition partial ordering. Define $Z(M)$ by*

$$Z(M) = s_1 \exp\left(-\sum_{i=1}^{\infty} \frac{s_i}{i} [Z(\mathcal{A})]\right),$$

where in $Z(\mathcal{A})$ the root point is not indicated. Suppose C and W are sets of graphs satisfying

$$Z(C) = Z(W)[Z(T)].$$

Then

$$Z(W) = Z(C)[Z(M)].$$

To prove this cycle index sum version, follow the proof of Theorem 2A. The crucial difference lies in the middle of the proof of Lemma 1. As stated, we were able to factor out the x weight from $Z(S_{l_m})[s_i \to -Z(A_m)[s_j \to x^{ij}]]$ which left $Z(S_{l_m})[s_j \to -1]$. This does not happen for cycle indices and so the answer appears only in the exponential form.

4. THE REFINEABLE CASE

We begin this section by considering a simple example of a refineable treelike set. Let T be the treelike set with unique atom

$\alpha =$  .

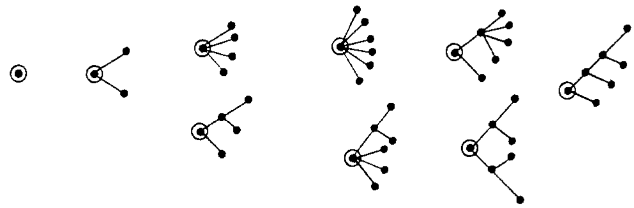


FIGURE 4

In this example our atom is built from 2 copies of the unrefineable atom $\odot$——$\bullet$. The set T through order 7 appears in Fig. 4. Let $C = T$ and $W = \{\odot\}$. Then $Z(C) = Z(T) = s_1[Z(T)] = Z(W)[Z(T)]$, and so C, T, and W satisfy Eq. (*) given in the hypothesis of Theorem 3. However Theorem 3 fails. To see this we need only consider permutations of 5 or fewer vertices represented by $Z(T)$. These terms are

$$Z(T) = s_1 + \tfrac{1}{2}s_1^3 + \tfrac{1}{2}s_1 s_2 + \tfrac{13}{24}s_1^5 + \tfrac{3}{4}s_1^3 s_2 + \tfrac{1}{3}s_1^2 s_3 + \tfrac{1}{8}s_1 s_2^2 + \tfrac{1}{4}s_1 s_4 + \cdots$$

There is one atom α and so $Z(\mathscr{A}) = \frac{1}{2}s_1^2 + \frac{1}{2}s_2$(the root point is not indicated in $Z(\mathscr{A})$). If Theorem 3 did hold we would have

$$s_1 = Z(W) = Z(T)\left[s_1 \exp\left(-\sum_{i=1}^{\infty} \frac{s_i[Z(\mathscr{A})]}{i}\right)\right]$$

or

$$s_1 = Z(T)[s_1 \exp(-\tfrac{1}{2}s_1^2 - \tfrac{1}{2}s_2 - \tfrac{1}{4}s_2^2 - \tfrac{1}{4}s_4 - \cdots)].$$

However, one finds that the right-hand side is equal to $s_1 - \frac{1}{12}s_1^5 - \frac{1}{4}s_1 s_2^2 + \frac{1}{3}s_1^2 s_3 + \cdots$. One will find that Lemma 1 also fails for the set T; the proof of Lemma 1 given in the last section breaks down at the point where unrefineability is used. Fortunately the main theorem remains true in the refineable case.

Let t be a treelike set of connected, labelled graphs. We partition t_∞ into a disjoint union of sets L_i, $i = 0, 1, 2,\ldots$, called *levels*. To construct the L_i, proceed by induction on i. Let L_0 be the set of all graphs in t_∞ having just one point. Given L_i, let $\bar{L}_{i+1}$ consist of all graphs α obtained from a graph $\beta \in L_i$ by replacing each vertex of β by either a 1-point graph or an atom in t_∞. Then let $L_{i+1} = \bar{L}_{i+1} - L_i$.

Given $\alpha \in \bar{L}_{i+1}$, let $\beta_1,\ldots,\beta_m$ be the graphs in L_i which are less than α. It is simple to check that one β_i is smallest in the ordering $(t_\infty, \leqslant)$; we call this β_i the *predecessor* of α.

THEOREM 2 (General Case). *Let T be a treelike set of rooted, connected graphs and let $\mathcal{A}$ be the set of atoms in its composition partial ordering.*

Define $M(x)$ by

$$M(x)=x\left(\prod_{A\in\mathcal{A}}(1-x^{|V(A)|-1})\right).$$

Suppose C and W are sets of graphs satisfying

$$Z(C)=Z(W)[Z(T)]. \tag{*}$$

Then

$$W(x)=Z(C)[M(x)].$$

Proof. The proof relies on many of the arguments utilized in the proof of Lemma 1. However the proof is somewhat harder and will only be sketched. Lemma 1 must be replaced by the following lemma.

LEMMA 2. $\lim_{n\to\infty}(1/n!)(\sum_{\sigma\in S_n}\sum_{w\in(t_n)_\sigma}Z(w;\sigma)[M(x)])=x$, *where $Z(w;\sigma)$ denotes the cycle indicator of σ acting on the set $V(w)$.*

Proof. Consider a graph w in t_n having more than one point and an automorphism $\sigma\in S_n$ of w. Suppose $w\in L_i$ for $i>1$, and let p be the predecessor to w in L_{i-1}. Let y_i for $i\in V(p)$ be such that w is obtained from p by vertex replacement with the graphs y_i. Let $V'(p)$ denote the subset of $V(p)$ consisting of those i such that y_i has more than one point. Note that σ is an automorphism of p and that σ acts on the set $V'(p)$.

Suppose $v\in t_n$ with $p\leqslant v\leqslant w$ and suppose that γ is an automorphism of v. Then γ is also an automorphism of p and so it makes sense to ask whether γ acts on the set $V'(p)$ and whether the action of γ agrees with the action of σ on $V'(p)$.

To prove Lemma 2 we proceed as follows. Isolate the one term of $Z(w;\sigma)[M(x)]$ of smallest degree, this being $x^{|w|}$ which comes from the lead term x in $M(x)$. We reorganize the sum

$$\frac{1}{n!}\left(\sum_{\sigma\in S_n}\sum_{w\in(t_n)_\sigma}Z(w;\sigma)[M(x)]\right) \tag{4.1}$$

so that this term $x^{|w|}$ is grouped with all terms $x^{|w|}$ which arise from $Z(v;\gamma)[M(x)]$, where $p\leqslant v\leqslant w$ and where γ restricted to $V'(p)$ agrees with σ restricted to $V'(p)$. One will find that the sum of all such terms is

$$x^{|w|}\left(\sum_{i=0}^{j_i}\binom{j_i}{i}(-1)^i\right)\cdots\left(\sum_{i=0}^{j_k}\binom{j_k}{i}(-1)^i\right), \tag{4.2}$$

where k is the number of cycles $C_1, \ldots, C_k$ in the disjoint cycle decomposition of σ acting on $V'(p)$ and where $j_l \geqslant 1$ is the number of isomorphism classes of atoms below y_m for any $m \in C_l$. Obviously expression (4.2) is 0.

Lastly one needs to argue that all terms x^d with $2 \leqslant d \leqslant n$ in (4.1) fall into one such group and so expression (4.1) is equal to $x + E_n(x)$, where $E_n(x)$ is a series in which every term has degree $n+1$ or higher. Lemma 2 follows at once. Return now to the proof of the theorem. From Lemma 2 it follows easily that $Z(T)[M(x)] = x$. So composing $M(x)$ over each side of (*) gives us the desired inversion. ∎

Consider the example given at the start of this section. Here $M(x) = x - x^3$ and $W(x) = x$. Up through order 7 we have

$$\begin{aligned}
Z(T)[M(x)] &= (s_1 + \tfrac{1}{2}s_1^3 + \tfrac{1}{2}s_1 s_2 + \tfrac{13}{24}s_1^5 + \tfrac{3}{4}s_1^3 s_2 \\
&\quad + \tfrac{1}{3}s_1^2 s_3 + \tfrac{1}{8}s_1 s_2^2 + \tfrac{1}{4}s_1 s_4)[x - x^3] + 5x^7 + \cdots \\
&= (x - x^3) + \tfrac{1}{2}(x - x^3)^3 + \tfrac{1}{2}(x - x^3)(x^2 - x^6) \\
&\quad + \tfrac{13}{24}(x - x^3)^5 + \tfrac{3}{4}(x - x^3)^3 (x^2 - x^6) + \tfrac{1}{3}(x - x^3)^2 (x^3 - x^9) \\
&\quad + \tfrac{1}{8}(x - x^3)(x^2 - x^6)^2 + \tfrac{1}{4}(x - x^3)(x^4 - x^{12}) + 5x^7 + \cdots \\
&= x + x^3(-1 + \tfrac{1}{2} + \tfrac{1}{2}) \\
&\quad + x^5(-\tfrac{3}{2} - \tfrac{1}{2} + \tfrac{13}{24} + \tfrac{3}{4} + \tfrac{1}{3} + \tfrac{1}{8} + \tfrac{1}{4}) \\
&\quad + x^7(\tfrac{3}{2} - \tfrac{1}{2} - \tfrac{65}{24} - \tfrac{9}{4} - \tfrac{2}{3} - \tfrac{1}{8} - \tfrac{1}{4} + 5) + \cdots \\
&= x.
\end{aligned}$$

5. Applications

Theorem 2 can be applied in the enumeration of graphs without endpoints, blocks, bridgeless graph, 3-connected graphs and 3-line connected graphs. In this section we demonstrate its use by considering its application in the first two cases. Both will follow as a corollary of the next theorem.

THEOREM 4. *Let B be a set of blocks, let E_B be the set of all connected graphs with two or more points having no endblocks in B and let C be the set of all connected graphs. Then*

$$E_B(x) = Z(C)\left[x \exp\left(-\sum_{i=1}^{\infty} \frac{B'(x^i)}{i}\right)\right] - x - B(x) + xB'(x).$$

Proof. Let F_B be the set of connected graphs all of whose blocks lie in B. Then

$$Z(C) = Z(F_B) + Z(E_B)[s_1 Z(F'_B)].$$

This equation can be established in exactly the same way that Robinson established the analogous equation for graphs without endpoints (see Robinson [2, p. 353]). It is easy to check that F'_B is a treelike set of rooted, connected graphs and that B' is the set of atoms in its composition ordering. By Theorem 2,

$$Z(C)\left[x\exp\left(-\sum_{i=1}^{\infty}\frac{B'(x^i)}{i}\right)\right]-Z(F_B)\left[x\exp\left(-\sum_{i=1}^{\infty}\frac{B'(x^i)}{i}\right)\right]=E_B(x). \tag{5.1}$$

By Robinson's Generalization of R. Z. Norman's techniques for counting graphs with given blocks (see Robinson [2, p. 348], and Norman [1]) we have

$$Z(F_B)=(s_1+Z(B)-s_1Z(B'))[s_1Z(F'_B)]. \tag{5.2}$$

Now, applying Theorem 2 again we have

$$Z(F_B)\left[x\exp\left(-\sum_{i=1}^{\infty}\frac{B'(x^i)}{i}\right)\right]=x+B(x)-xB'(x). \tag{5.3}$$

Substituting (5.3) in (5.1) gives the desired result.

COROLLARY 4.1 (R. W. Robinson). *Let G be the set of all graphs and let W be the set of all graphs without endpoints. Then*

$$W(x)=Z(G)[x-x^2].$$

Proof. Let B be the one element set of blocks containing only the block with 2 points; i.e., $B=\{\bullet\!\!-\!\!-\!\!-\!\!\bullet\}$. W is precisely the set of graphs without end blocks in B. Let W^c denote the set of connected graphs in W. By Theorem 3, $W^c(x)-x=Z(C)[x\exp(-\sum_{i=1}^{\infty}(x^i/i))]-x-x^2+x^2$ and so

$$W^c(x)=Z(C)[x-x^2]. \quad \blacksquare$$

COROLLARY 4.2 (R. W. Robinson). *Let H be the set of all nonseparable graphs and let $A(x)$ be the unique generating function which satisfies*

$$x=A(x)\,Z(C')[A(x)].$$

Then

$$xH'(x)=x\sum_{j=1}^{\infty}\frac{u(j)}{j}\log\left(\frac{A(x^j)}{x^j}\right)$$

TABLE I

Unlabelled Graphs without Endpoints

n	w_n
1	1
2	1
3	2
4	5
5	16
6	78
7	588
8	8047
9	205914
10	10014882
11	912908876
12	154636289460
13	48597794716736
14	28412296651708628
15	31024938435794151088
16	63533059372622888758054
17	244916078509480823407040988
18	1783406527599529094009748567708
19	246056746234744284]5849066062642456
20	64502234208884158207776560016341075 0816

and

$$H(x) = Z(C)[A(x)] + x\left(\sum_{j=1}^{\infty} \frac{u(j)}{j} \log\left(\frac{A(x^j)}{x^j}\right) - 1\right).$$

Proof. Let $B = H$ in Theorem 3. Then $E_B = \varnothing$ and so

$$0 = Z(C)\left[x \exp\left(-\sum_{i=1}^{\infty} \frac{H'(x^i)}{i}\right)\right] + xH'(x) - H(x) - x.$$

Hence

$$H(x) = Z(C)\left[x \exp\left(-\sum_{i=1}^{\infty} \frac{H'(x^i)}{i}\right)\right] + x(H'(x) - 1).$$

Let

$$A(x) = x \exp\left(-\sum_{i=1}^{\infty} \frac{H'(x^i)}{i}\right).$$

It is straightforward to show that

$$x(H'(x)-1)=x\left(\sum_{j=1}^{\infty}\frac{\mu(j)}{j}\log\left(\frac{A(x^j)}{x^j}\right)-1\right),$$

and so it suffices to show that $A(x)$ satisfies the recursion

$$x=A(x)\,Z(C')[A(x)]=(s_1Z(C'))[A(x)]. \tag{5.4}$$

To prove (5.4), we will prove its cycle index form. Let

$$A(s_1,s_2,\ldots)=s_1\exp\left(-\sum_{i=1}^{\infty}\frac{s_i}{i}\,[Z(H')]\right).$$

Instead of proving (5.4), we will prove that

$$s_1=(s_1Z(C'))[A(s_1,s_2,\ldots)].$$

To do so we will need the following lemma.

LEMMA 3. *Let* $S(s_1,s_2,\ldots)$, $T(s_1,s_2,\ldots)\in\mathbb{Q}[[s_1,s_2,\ldots]]$, *and suppose both* S *and* T *have zero constant term. If*

$$s_1=S(s_1,s_2,\ldots)[T(s_1,s_2,\ldots)]$$

then

$$s_1=T(s_1,s_2,\ldots)[S(s_1,s_2,\ldots)].$$

Proof. Consider the $\mathbb{Q}$-algebra homomorphism $\varphi\colon\mathbb{Q}[[s_1,s_2,\ldots]]\to\mathbb{Q}[[s_1,s_2,\ldots]]$ given by

$$\varphi(s_i)=s_i[T(s_1,s_2,\ldots)].$$

As $s_1=S(s_1,s_2,\ldots)[T(s_1,s_2,\ldots)]$, T must have the form $T(s_1,s_2,\ldots)=qs_1+T_2(s_1,s_2,\ldots)$, where q is a nonzero element of $\mathbb{Q}$ and where $T_2(s_1,s_2,\ldots)$ consists only of terms having degree two or more (the *degree* of a term $rs_1^{a_i}\cdots s_n^{a_n}$ is $\sum ia_i$). Since T has this form, φ is an isomorphism. Now

$$\begin{aligned}
&\varphi(T(s_1,s_2,\ldots)[S(s_1,s_2,\ldots)])\\
&\quad=T(s_1,\ldots)[(S(s_1,s_2,\ldots))[T(s_1,\ldots)]]\\
&\quad=T(s_1,s_2,\ldots)[s_1]\\
&\quad=T(s_1,s_2,\ldots)\\
&\quad=s_1[T(s_1,s_2,\ldots)]\\
&\quad=\varphi(s_1).
\end{aligned}$$

TABLE II

Unlabelled Nonseparable Graphs

n	h_n
1	0
2	1
3	1
4	3
5	10
6	56
7	468
8	7123
9	194066
10	9743542
11	900969091
12	153620333545
13	48432939159704
14	28361824488394169
15	30995890806033380784
16	63501635429109597504951
17	244852079292073376010411280
18	1783160594069429925952824734641
19	24603887051350945867492816663958981
20	644997704304598761531891390989833304810

Since φ is 1–1, the result follows, which completes the proof of Lemma 2.

We now return to the proof of Corollary 4.2. In view of Lemma 3, it is clear that proving (4.5) is equivalent to proving

$$s_1 = A(s_1, s_2, \ldots)[s_1 Z(C')]. \tag{5.6}$$

R. W. Robinson (see [2, p. 349]) proved that

$$Z(C') = \left(\exp\left(\sum_{i=1}^{\infty} \frac{s_i}{i}\,[Z(H')]\right)[s_1 Z(C')]\right)$$

and so,

$$A(s_1, s_2, \ldots)[s_1 Z(C')] = s_1,$$

as was to be shown. ∎

Tables I and II were computed from the equations given in Corollaries 4.1 and 4.2 by Albert Nymeyer of the University of Newcastle, New South Wales, Australia. In both cases, his computations have gone as far as

$n = 26$. He did this work in the course of research on a project directed by R. W. Robinson and sponsored by the Australian Research Grants Committee. Thanks are due to both Mr. Nymeyer and Professor Robinson for permitting me to use the tables.

Corollaries 4.1 and 4.2 show that two different things can happen when Theorem 2 is applied. In the case of graphs without endpoints, the inverting series $M(x)$ is easily produced. In the case of nonseparable graphs, the inverting series $M(x)$ is no easier to produce than the desired series $W(x)$. But in this case, Theorem 2 tells us that $M(x)$ (as well as $W(x)$) counts something of interest. Even though there is some work involved in computing $M(x)$, once we have it we have counted both nonseparable and rooted nonseparable graphs.

4. Conclusion

It is interesting to observe that when Theorem 2 is stated in its generating function form, the power series $M(x)$ happens to be

$$M(x) = \sum_{A \in T} \mu(0, A)\, x^{|V(A)|},$$

where μ is the Möbius function in the incidence algebra of the composition partial ordering $(T, \leqq)$.

Such a coincidence suggests that perhaps Theorem 2 can be proven in its generating function form directly within the incidence algebra of $(T, \leqq)$ and with no mention of the fixed point partial orderings. This would avoid the laborious counting arguments which are used in the proofs Lemma 1 and Theorem 2. Note that the analogous thing does *not* happen when Theorem 2 is stated in its cycle index sum form; i.e.,

$$s_1 \exp\left(-\sum_{i=1}^{\infty} \frac{s_i}{i} [Z(\mathcal{O})]\right) \neq \sum_{A \in T} \mu(0, A)\, Z(A).$$

In summary, Theorem 2 provides a combinatorial interpretation for the generating function $M(x)$ which inverts (*) to (***). Besides those listed here, Theorem 2 has applications in the enumeration of bridgeless graphs, 3-connected graphs and 3-line connected graphs. In each of these latter cases, one uses Theorem 2 to obtain equations very similar to those obtained in Corollary 3.2.

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